Littlewood's First Principle (cf. Th 3, 4 & Ex in §3, & Th1 in §4 of Notes). Let $E \subseteq \mathbb{R}$. Then $\circlearrowleft$:

I. (i) $E \in \mathfrak{M}$ (E is measurable).

(ii) E is outer-regular: $\forall \varepsilon > 0$ $\exists$ open $O \supseteq E$ s.t. $m^*(O \setminus E) < \varepsilon$.

(iii) $\exists$ a G_δ-set $G \supseteq E$ s.t. $m^*(G \setminus E) = 0$.

(iv) E is inner regular: $\forall \varepsilon > 0$ $\exists$ closed $F \subset E$ s.t. $m^*(E \setminus F) < \varepsilon$.

(v) $\exists$ a F_σ-set $H \subset E$ s.t. $m^*(E \setminus H) = 0$.

II. Assume further that $m^*(E) < +\infty$. Then each of (i) – (v) is equivalent to

(vi) $\forall \varepsilon > 0$. $\exists$ disjoint open intervals $I_1, I_2, \cdots, I_n$ of finite lengths (with some $n \in \mathbb{N}$) such that $m^*(E \triangle \bigcup_{i=1}^{n} I_i) < \varepsilon$.

III. Let $m(E) < +\infty$. Then $\forall \varepsilon > 0$ $\exists$ a step-function $g : \mathbb{R} \to \mathbb{R}$ (& "supported by" a finite (length) interval J in the sense that $g = 0$ on $\mathbb{R} \setminus J$) such that

$\chi_E(x) = g(x)$ $\forall x \in \mathbb{R} \setminus A$ with some A of measure $< \varepsilon$.

IV. Let $m(E) < +\infty$ and let $f : E \to \mathbb{R}$ be a "simple function" on E ($f = \sum_{j=1}^{n} a_j \chi_{E_j}$, $a_j \in \mathbb{R}$, $E_j \in \mathfrak{M}$ & $E_j \subseteq E$ $\forall i$). Then $\forall \varepsilon > 0$ $\exists$ a step-function $g : \mathbb{R} \to \mathbb{R}$ supported by a finite interval such that $f(x) = g(x)$ $\forall x \in E \setminus A$ with some A of mea $< \varepsilon$

V (Corollary of the First Principle of Littlewood. Let $f \in \mathcal{L}$ supported by E with $m(E) < +\infty$. Then $\forall \varepsilon > 0$ $\exists$ $\{$step-function g, continuous function h supported by a finite length) interval such that $f = g = h$ on $E \setminus A$ with some A of measure $< \varepsilon$.

Proof. In light of IV, we need only show how to approximate a step-function $g : [a,b] \to \mathbb{R}$. It is done by removing the (finitely, say N, many) (jump)-discontinuity points of g: around each discontinuity z_i, the graph of g on $\left[z_i - \frac{\varepsilon}{2N}, z_i + \frac{\varepsilon}{2N}\right]$ is replaced by the line-segment joining points $\left(z_i - \frac{\varepsilon}{2N}, g\left(z_i - \frac{\varepsilon}{2N}\right)\right)$ and $\left(z_i + \frac{\varepsilon}{2N}, g\left(z_i + \frac{\varepsilon}{2N}\right)\right)$. Let h denote the piecewise-linear function so obtain. Then, as the total length of these intervals is $\frac{\varepsilon}{N} \times N = \varepsilon$, we see that $g = h$ on $E \setminus A$ with $m(A) \leq \varepsilon$, and that h is supported by a finite interval.

Revisit for the proof of (vi) $\Rightarrow$ (ii). Let $\varepsilon > 0$. By (vi), $\exists$ open set U s.t. $m^*(U \triangle E) < \varepsilon/2$. Since LHS equals $\inf\{m(O) : \text{open } O \supseteq U \triangle E\}$, take open $O \supseteq E \setminus U$ of measure $< \varepsilon/2$. Then $O \cup U$ covers E and

$$(O \cup U) \setminus E \subseteq O \cup (U \setminus E) \text{ of mea} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

so $O \cup U$ is a desired open set for (ii).

2nd Principle. Let $f \in \mathcal{MF}(E)$ with $m(E) < +\infty$ and $f(x) \in \mathbb{R}$ a.e. x in E. Then, $\forall \varepsilon > 0$, $\exists$ a "nice-function" φ ("nice" can mean any of $\left\{\begin{matrix}\text{simple}\\ \text{step}\\ \text{continuous}\end{matrix}\right\}$) supported by a finite interval such that $|f - \varphi| < \varepsilon$ on $E \setminus A$ with some A of mea $< \varepsilon$.

Proof By the preceding corollary of the 1st principle, we need only consider the assertion regarding simple function φ. We do this in two parts.

I. Assume in addition that f is bounded and $E \subseteq [a,b]$, so $\exists$ $m < M$ in $\mathbb{R}$ s.t. $f : E \longrightarrow [m, M)$. Take $N \in \mathbb{N}$ s.t. $\frac{M-m}{N} < \varepsilon$ and divide $[m,M)$ into N-many subintervals of equal length by partition points $\{m = y_0 < y_1 < \cdots < y_{N-1} < y_N = M\}$. For $i = 1, 2, \cdots N$, let

$$Z_i := E \cap f^{-1}\big([y_{i-1}, y_i)\big)$$

(so measurable)

and

$$\varphi := \sum_{i=1}^{N} y_{i-1} \chi_{Z_i}$$

Then $\varphi \in \mathcal{S}$, supported by $E \subseteq [a,b]$ and $0 \leq f - \varphi \leq \max_{1 \leq i \leq N} (y_i - y_{i-1}) < \varepsilon$ on E

of the proof for the 2nd Principle

II. The general case (as stated in the 2nd Principle).

For each $n \in \mathbb{N}$, let

$$E_n = \{ x \in E \cap [-n, n] : |f(x)| < n \}$$

Then $E_n \uparrow_n \bigcup_{n \in \mathbb{N}} E_n$ (of measure $= m(E) < +\infty$)

so $m(E_n) \uparrow_n m(E)$ and $\exists\, n \in \mathbb{N}$ s.t.

$m(E_n) > m(E) - \varepsilon$, i.e. $A := E \setminus E_n$ is of measure $< \varepsilon$, and f is bounded on $\underbrace{E \setminus A}_{= E_n \subseteq [-n, n]}$.

By part I, $\exists\, \varphi \in \mathcal{S}$ supported by $E \setminus A$ s.t.

$$0 \leq f - \varphi < \varepsilon \text{ on } E \setminus A,$$

3rd Principle (Egoroff Theorem): Let $\mathcal{MF}(E) \ni f_n \to f$ a.e. on E 5

and $f(x) \in \mathbb{R}$ a.e. x in E. Suppose $m(E) < +\infty$.

Then, $\forall \eta > 0$, $\exists$ A of mea $< \eta$ s.t. $f_n \to f$ uniformly on $E \setminus A$.

Proof. I. Let $\varepsilon, \delta > 0$. Then $\exists$ $A_{\varepsilon,\delta}$ ~~$A(\varepsilon,\delta)$~~ of mea $< \delta$ and $N_{\varepsilon,\delta} \in \mathbb{N}$ such that $|f_n(\cdot) - f(\cdot)| < \varepsilon$ on $E \setminus A_{\varepsilon,\delta}$

$\forall$ ~~$n \geq N(\varepsilon,\delta)$~~ $n \geq N_{\varepsilon,\delta}$. Indeed,

$$\{x \in E : \mathbb{R} \ni f(x) = \lim_n f_n(x)\} = \bigcap_{\varepsilon > 0} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{x \in E : |f_n(x) - f(x)| < \varepsilon\}$$

(so is of measure $= m(E) < +\infty$); consequently

$$\bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{x \in E : |f_n(x) - f(x)| < \varepsilon\}$$

is of mea $= m(E)$; since

$$E_N := \bigcap_{n \geq N} \{x \in E : |f_n(x) - f(x)| < \varepsilon\} \uparrow \bigcup_{N \in \mathbb{N}} E_N$$

it follows that

$$m(E_N) \uparrow_N m(E)$$

Take $N_{\varepsilon,\delta} \in \mathbb{N}$ s.t. $A_{\varepsilon,\delta} := E \setminus E_N$ is of mea $< \delta$

Then $E \setminus A_{\varepsilon,\delta} = E_{N_{\varepsilon,\delta}}$ and so

$$|f_n(x) - f(x)| < \varepsilon \quad \forall x \in E \setminus A_{\varepsilon,\delta} \;\&\; \forall n \geq N_{\varepsilon,\delta}$$

II. of the proof for 3rd Principle

Apply I to $(\varepsilon, \delta) = \left(\frac{1}{q}, \frac{\eta}{2^q}\right)$ ~~and~~

we have A_q of mea $< \frac{\eta}{2^q}$ and

$N_q \in \mathbb{N}$ s.t.

$$|f_n - f| < \frac{1}{q} \text{ on } E \setminus A_q, \ \forall n \geq N_q$$

Let $A = \bigcup_{q \in \mathbb{N}} A_q$ (of mea $< \eta$). Then

$$|f_n - f| < \frac{1}{q} \text{ on } E \setminus A, \ \forall n \geq N_q ;$$

thus $f_n \to f$ uniformly on $E \setminus A$.